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An equatorial boundary layer

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The singularity of the Ekman layer at the equator of a rotating gravitating sphere makes it difficult to satisfy a prescribed stress boundary condition at the surface of a layer of liquid on the sphere. The equations of motion are investigated for a homogeneous ocean with vertical and lateral eddy viscosities. The horizontal Coriolis terms are not neglected. A linear equation for the boundary layer is obtained and a solution of the equation for the boundary-layer part of the velocity field is found in closed form. This is valid in a parameter range which includes the previous solutions of Stewartson and Gill as limiting cases.

1. Introduction

Investigations into the properties of the flow of a liquid on the surface of a rotating sphere, whether abstract studies such as that of Proudman (1956) or studies whose motivation is an observational science such as oceanography, usually show a change of character near to the sphere's equator. In this region the vertical component of the rotation vector becomes small, so that the thickness of the Ekman layer increases indefinitely and the boundary-layer approximation fails.

The equatorial region has been investigated under a number of different conditions by several authors, notably Stommel (1960), Carrier (1965), Stewartson (1966) and Gill (1971). If a critical depth

$$h_c = (\nu^2 R/4\Omega^2)^{\frac{1}{5}}$$

is defined, where ν is the vertical component of eddy viscosity, R is the radius of the earth and Ω is its rate of rotation, the ratio of h_c to the thickness h of the layer of liquid can be used to relate these models as follows.

Stommel's model. Viscous effects are important at all depths of the fluid near the equator and the horizontal Coriolis terms are important if $h_c \gg h$.

Carrier's model. When $h \sim h_c$ the zeroth-order flow is the same as in Stommel's model but the horizontal Coriolis terms affect the first-order correction to the zeroth-order flow.

Stewartson's model. When $h \gg h_c$ viscous effects are confined to a thin surface boundary layer and the horizontal Coriolis terms are crucial.

These three models all neglect lateral friction. Gill's model is similar to Stommel's in that the horizontal Coriolos term is neglected, but lateral friction is included and the model shows that although the effects of vertical friction are confined to a surface boundary layer the effects of lateral friction are important at all depths.

The purpose of this paper is to investigate further the linearized equations of motion on the assumption that $h \gg h_c$. The surface stress will be specified, rather than the velocity, as this is the condition that applies to the oceans. The effects of lateral friction are included so that it is possible to study the transition from Stewartson's model to Gill's; at general latitudes it can be ignored but its importance increases near the equator as Gill has demonstrated. A solution of boundary-layer form can be found for all values of the coefficient of lateral friction, and it is possible to obtain an explicit solution for the boundary layer calculated by Gill as the limiting case in which the effects of lateral friction entirely mask the horizontal component of the Coriolis force.

2. The boundary-layer equation

In a region near to the equator in which no variation with longitude is expected and $h \ll R$, the equations of motion for constant density and constant vertical and horizontal eddy viscosities ν and η can be written approximately as

$$u = \frac{1}{h}\frac{\partial \chi}{\partial z'}, \quad v = \frac{1}{h}\frac{\partial \psi}{\partial z'}, \quad w = -\frac{1}{R}\frac{\partial \psi}{\partial \phi}, \tag{2.1}$$

$$-2\frac{\Omega}{h}\frac{\partial \chi}{\partial z'} = -g - \frac{1}{\rho h}\frac{\partial p}{\partial z'}, \tag{2.2}$$

$$2\frac{\Omega}{h}\frac{\partial \chi}{\partial z'}\phi = -\frac{1}{R\rho}\frac{\partial p}{\partial \phi} + \frac{\nu}{h^3}\frac{\partial^3 \psi}{\partial z'^3} + \frac{\eta}{hR^2}\frac{\partial^3 \psi}{\partial \phi^2 \partial z'} \tag{2.3}$$

and

$$-\frac{2\Omega}{h}\frac{\partial \psi}{\partial z'}\phi - \frac{2\Omega}{R}\frac{\partial \psi}{\partial \phi} = \frac{\nu}{h^3}\frac{\partial^3 \chi}{\partial z'^3} + \frac{\eta}{hR^2}\frac{\partial^3 \chi}{\partial \phi^2 \partial z'}. \tag{2.4}$$

The terms neglected, which include the inertia terms, are expected to be smaller than the smallest terms retained, and this can be checked after a solution has been found. Here, u, v and w are the velocity components in the directions East, North and vertically up, respectively; the co-ordinates are latitude ϕ (which is assumed to be small) and z', which has been scaled by the depth of the ocean. The function ψ has been introduced to satisfy the continuity condition, while χ has been introduced to provide a symmetric notation. The symbols g, Ω and p stand for the acceleration due to gravity, the magnitude of the earth's rotation vector and pressure at a point in the fluid, respectively.

If

$$N' = \chi + i\psi$$

(2.2)–(2.4) may be combined to give

$$\frac{\nu}{2\Omega h^2}\frac{\partial^3 N'}{\partial z'^3} + \frac{\eta}{2\Omega R^2}\frac{\partial^3 N'}{\partial \phi^2 \partial z'} = i\left(\phi\frac{\partial N'}{\partial z'} + \frac{h}{R}\frac{\partial N'}{\partial \phi}\right) + iQ(\phi), \tag{2.5}$$

where $Q(\phi)$ is a function of position only. This equation is similar to those produced by Stewartson (1966) and Carrier (1965).

To study this equation further, N' can be rewritten as

$$N' = N^I + N,$$

where N^I is the part of N' which is non-zero in the interior of the ocean, where effects due to the vertical component of eddy viscosity are negligible. This contribution can be found by the methods used by Gill (1971), for example. N is then the part of N' which vanishes outside a boundary layer at the surface. A similar contribution can be added to N' for the bottom boundary layer if this has to be considered also. The function N then satisfies an equation which is the same as (2.5) but without the final term $iQ(\phi)$.

To investigate the orders of magnitude of the various terms, introduce the following dimensionless numbers:

$$r_1 = \nu/2\Omega h^2, \quad r_2 = h/R, \quad r_3 = \eta/2\Omega R^2;$$

now suppose that variation with respect to z' takes place over a distance of order δ, and that with respect to ϕ over a range of latitude of order ϵ. The orders of the terms in (2.5) are in the ratio

$$\frac{r_1}{\delta^3} : \frac{r_3}{\delta\epsilon^2} : \frac{\epsilon}{\delta} : \frac{r_2}{\epsilon}.$$

In a surface boundary layer, ϵ is a measure of distance from the equator and so an inspection of this set of ratios will show which terms dominate in the surface layer as the equator is approached. The horizontal eddy viscosity which appears in r_3 can be retained as a parameter which may be given any desired value.

(i) When $\epsilon^{\frac{5}{2}} \gg r_1^{\frac{1}{2}} r_2$, a layer for which $\delta \sim r_1^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}$ is obtained. When the effects of horizontal eddy viscosity are neglected, this is the Ekman layer.

(ii) When $\epsilon = O(r_1^{\frac{1}{5}} r_2^{\frac{2}{5}})$, a layer for which $\delta \sim r_1^{\frac{2}{5}} r_2^{-\frac{1}{5}} = h_c/h$ results, and all terms of the equation are of importance, with the possible exception of the term involving r_3.

In the first case the effects of horizontal eddy viscosity are not of importance in most applications, but in the second a boundary layer can arise which is dominated by the horizontal eddy viscosity rather than by the horizontal component of the rotation vector. This occurs if $r_3^{\frac{1}{3}} \gg r_1^{\frac{1}{5}} r_2^{\frac{2}{5}}$. It requires a relatively large value of the horizontal eddy viscosity and a relatively small value of the vertical eddy viscosity to satisfy this inequality, so it seems that, although this third possibility may be of some interest, it is preferable to study the problem with both effects included. The two extreme cases can then be investigated from the solution obtained.

With the substitutions

$$z' = \frac{R^{\frac{1}{5}}}{h}\left(\frac{\nu}{2\Omega}\right)^{\frac{2}{5}} z, \quad \phi = \frac{1}{R^{\frac{2}{5}}}\left(\frac{\nu}{2\Omega}\right)^{\frac{1}{5}} y, \quad r = \frac{\eta}{\nu^{\frac{3}{5}}(2\Omega R^2)^{\frac{2}{5}}},$$

equation (2.5) becomes

$$\frac{\partial^3 N}{\partial z^3} + r\frac{\partial^3 N}{\partial z\,\partial y^2} = i\left(y\frac{\partial N}{\partial z} + \frac{\partial N}{\partial y}\right). \tag{2.6}$$

3. The surface boundary layer

If the ocean is driven by a wind stress which is symmetric about the equator and does not have a large horizontal gradient there, then N can be scaled so that the boundary conditions for which (2.6) has to be solved are the following.

$$(a) \qquad \partial^2 N/\partial z^2 \to -1 \quad \text{as} \quad z \to 0^-. \tag{3.1}$$

$$(b) \qquad N \to 0 \quad \text{as} \quad z \to -\infty, \tag{3.2}$$

since N gives that part of the velocity field which vanishes outside the boundary layer,

$$(c) \qquad \frac{\partial N}{\partial z} \sim \frac{-1 \pm i}{(2|y|)^{\frac{1}{2}}} \exp\left[z(1 \pm i)\left(\tfrac{1}{2}|y|\right)^{\frac{1}{2}}\right] \quad \text{as} \quad y \to \pm\infty. \tag{3.3}$$

(d) The vertical velocity vanishes at the surface; since conditions (a), (b) and (c) together with (2.6) determine the boundary-layer part of the vertical velocity, this implies a condition on the velocity outside the boundary layer.

The form of (2.6) suggests the solution

$$N = \int_C F(y,k)\, e^{kz}\, dk,$$

where C is a contour in the complex-k plane. Condition (b) suggests that one restriction on C will be

$$\operatorname{Re}(k) \geqslant 0 \quad \text{on} \quad C. \tag{3.4}$$

The kernel F then satisfies the equation

$$rk\frac{\partial^2 F}{\partial y^2} - i\frac{\partial F}{\partial y} + (k^3 - iyk)F = B(k),$$

where

$$\int_C B(k)\, e^{kz}\, dk = 0. \tag{3.5}$$

Thus

$$F = \int_D \frac{B(k)}{k} \exp\left(-\tfrac{1}{3}rp^3 - p^2/2k + k^2 p - ipy\right) dp$$

provided that D is chosen so that

$$\left[\exp\left(-\tfrac{1}{3}rp^3 - p^2/2k + k^2 p - ipy\right)\right]_D = 1.$$

Since N is required to be bounded for large $|y|$ the appropriate path for D is the positive real axis. When r is non-zero the integral for F converges whatever contour C is chosen, otherwise condition (3.4) ensures convergence provided that

$$\operatorname{Re}(k^2) < 0 \quad \text{when} \quad \operatorname{Re}(k) = 0.$$

Consider first the asymptotic behaviour of N for large values of $|y|$. The second derivative of N with respect to z is

$$\int_C \int_{p=0}^{\infty} kB(k) \exp\left(-\tfrac{1}{3}rp^3 - p^2/2k + k^2 p - ipy + kz\right) dp\, dk,$$

which is asymptotically equal to

$$\int_C \frac{kB(k)e^{kz}}{iy-k^2}dk$$

if C satisfies $\mathrm{Re}\,(k^2) > 0$.

It later proves to be impossible to choose a contour which satisfies this condition on the whole of its length, but it is only necessary for it to be possible to deform C so that this condition is satisfied on all portions of C except those which give an arbitrarily small contribution, and this is easily shown to be possible.

If C is chosen to be a straight line parallel to the imaginary axis in $\mathrm{Re}\,(k) > 0$, these conditions are all satisfied, and the contour can be deformed to show that

$$\partial^2 N/dz^2 \sim \pi B[(iy)^{\frac{1}{2}}]\exp[z(iy)^{\frac{1}{2}}] \quad \text{as} \quad |y|\to\infty,$$

where the square root taken has a positive real part. If B is chosen to be $-1/\pi$ the asymptotic conditions are then satisfied, and N has the form

$$\frac{i}{\pi}\int_C\int_{p=0}^{\infty}\frac{1}{k}\exp(-\tfrac{1}{3}rp^3-p^2/2k+k^2p-ipy+kz)\,dp\,dk. \tag{3.6}$$

The expression for N_{zz} can be written as

$$-\frac{1}{\pi}\int_0^{\infty}\frac{1}{p^{\frac{3}{2}}}\exp(-\tfrac{1}{3}rp^3-ipy-z^2/4p)\int_{-\infty}^{\infty}(ik\sqrt{p}-\tfrac{1}{2}z)\exp\left(-k^2-\frac{p^3}{2ik\sqrt{p}-z}\right)dk\,dp.$$

The inner integral is absolutely and uniformly convergent for all positive and zero values of p and $-z$. Scaling p with z^2 then shows that the expression tends to -1 as z tends to zero from below, and hence the surface boundary condition (3.1) is satisfied for all values of y. Since the expression tends to zero as $|z|\to\infty$, the solution is of boundary-layer type.

Expression (3.6) can be integrated numerically and the velocity components compared with the corresponding Ekman layer (3.3). This comparison is shown in figure 1 for the special case $r = 0$, in which the effects of the horizontal component of the eddy viscosity are neglected. In figure 2 the comparison is shown for the corresponding solution of the equation

$$\frac{\partial^3 N}{\partial y^2\partial z}+\frac{\partial^3 N}{\partial z^3} = iy\frac{\partial N}{\partial z}. \tag{3.7}$$

This represents the other extreme example in which the horizontal eddy viscosity is assumed to be much more important than the horizontal component of the rotation vector. The solution of this equation is

$$\frac{\partial N}{\partial z} = -\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{1}{\sqrt{p}}\exp(-\tfrac{1}{3}p^3-ipy-z^2/4p)\,dp. \tag{3.8}$$

The scaling of y' and z' is different here, and is chosen for direct comparison with the Ekman layer given in (3.3). This boundary layer can be superimposed on Gill's solution for the equatorial undercurrent in order to satisfy the surface conditions. In particular, it can be shown that this combination satisfies condition (*d*).

The solution of (3.7) has been found numerically by D. McKee and is shown in Gill's paper superimposed on the solution he found for the region of the ocean far

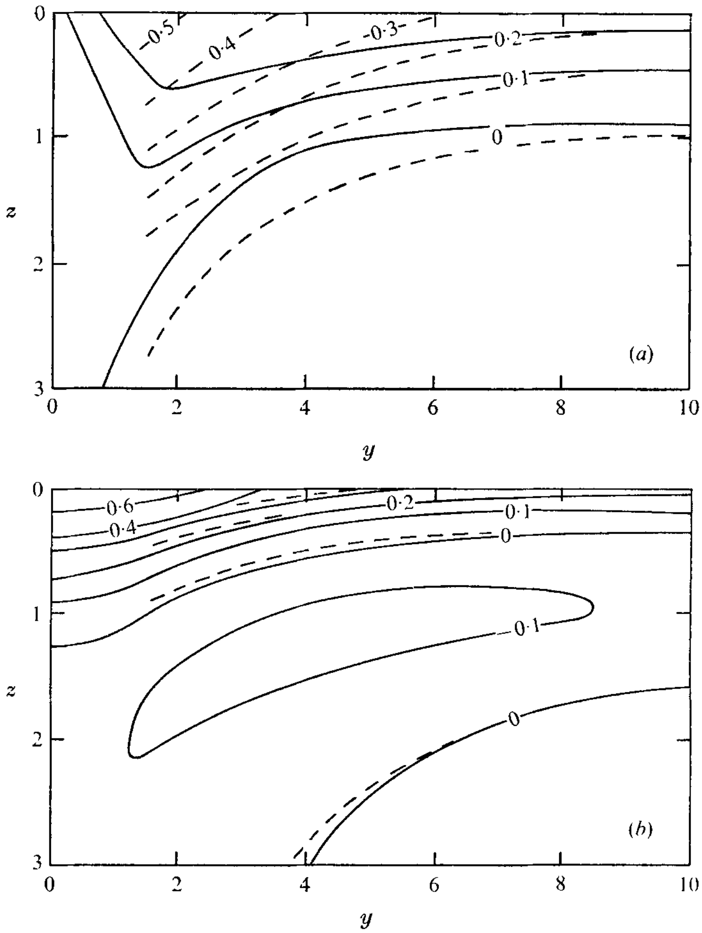

FIGURE 1. Contours of velocity components for unit surface stress as functions of y and z neglecting horizontal eddy viscosity. The broken lines indicate contours for the corresponding Ekman layer. Here, $z = (4\Omega^2/R\nu^2)^{\frac{1}{5}}d$ and $y = (2\Omega R^2/\nu)^{\frac{1}{5}}\phi$, where d is depth from the surface and ϕ is the angle of latitude. (*a*) Velocity component to the North. (*b*) Velocity component to the West.

from the surface. Like the numerical solution in Gill's paper, figure 2 shows that there is a strong westward velocity in the boundary layer, and that it falls off with depth and with increasing latitude. Reverse flow occurs at a depth which increases as the equator is approached, but there is no eastward flow at the equator. The northward velocity component at the surface increases from zero to a maximum from which it falls as the solution becomes more like the Ekman layer. The upwelling into the boundary layer is given by Gill: an alternative expression for it can be found from (3.8) and is

$$\int_0^\infty pe^{-\frac{1}{5}p^5}\cos py\, dp;$$

this gives an upwelling near to the equator but farther away there is a downward flux with velocity approximately equal to $1/y^2$.

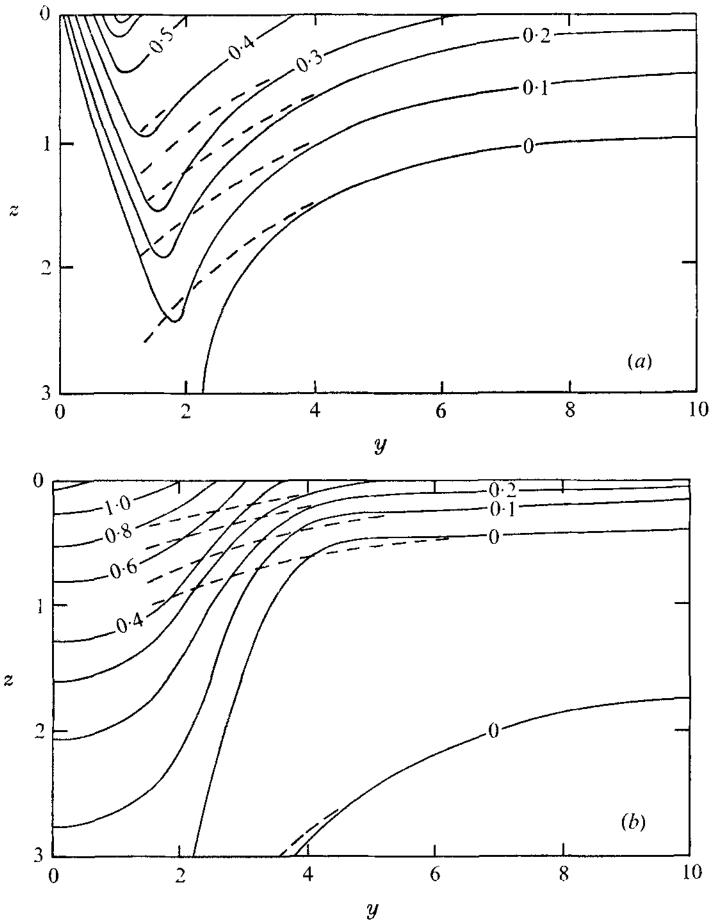

FIGURE 2. Contours of velocity components for unit surface stress as functions of y and z with horizontal eddy viscosity but neglecting the horizontal component of the rotation vector. Here, $z = (4\Omega^2\eta/R^2\nu^3)^{\frac{1}{6}}d$ and $y = (2\Omega R^2/\eta)^{\frac{1}{3}}\phi$, where d is depth from the surface and ϕ is the angle of latitude. The broken lines indicate contours for the corresponding Ekman layer. (Note that η appears in the scaling of y as well as of z, in such a way that the dimensional Ekman thickness for large values of y is independent of η, as it should be.) (*a*) Velocity component to the North. (*b*) Velocity component to the West.

Equation (2.6) has been solved numerically with r equal to zero by Philander (1971), who used a constant-velocity surface condition in the meridional direction. The solution shown in figure 1 has the same qualitative features as that shown in figure 2: the main difference is that the region of reverse flow in the westward velocity field reaches to the equator. This can be compared with the same feature which appears in the northward velocity component of Philander's solution.

These figures illustrate the fact that the velocity component decreases rapidly away from the surface and that the boundary layer is not singular at the equator. In this last property they are therefore quite different from the Ekman layer.

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